Expected shortfall of claims amounts: some practical aspects

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Introduction

In insurance one may be confronted to the expected shortfall estimation when one has:

- to price an XS or Stop-Loss reinsurance treaty;
- to calibrate the required capital in the Swiss Solvency Test (99% expected shortfall);
- more generally, to measure the risk.

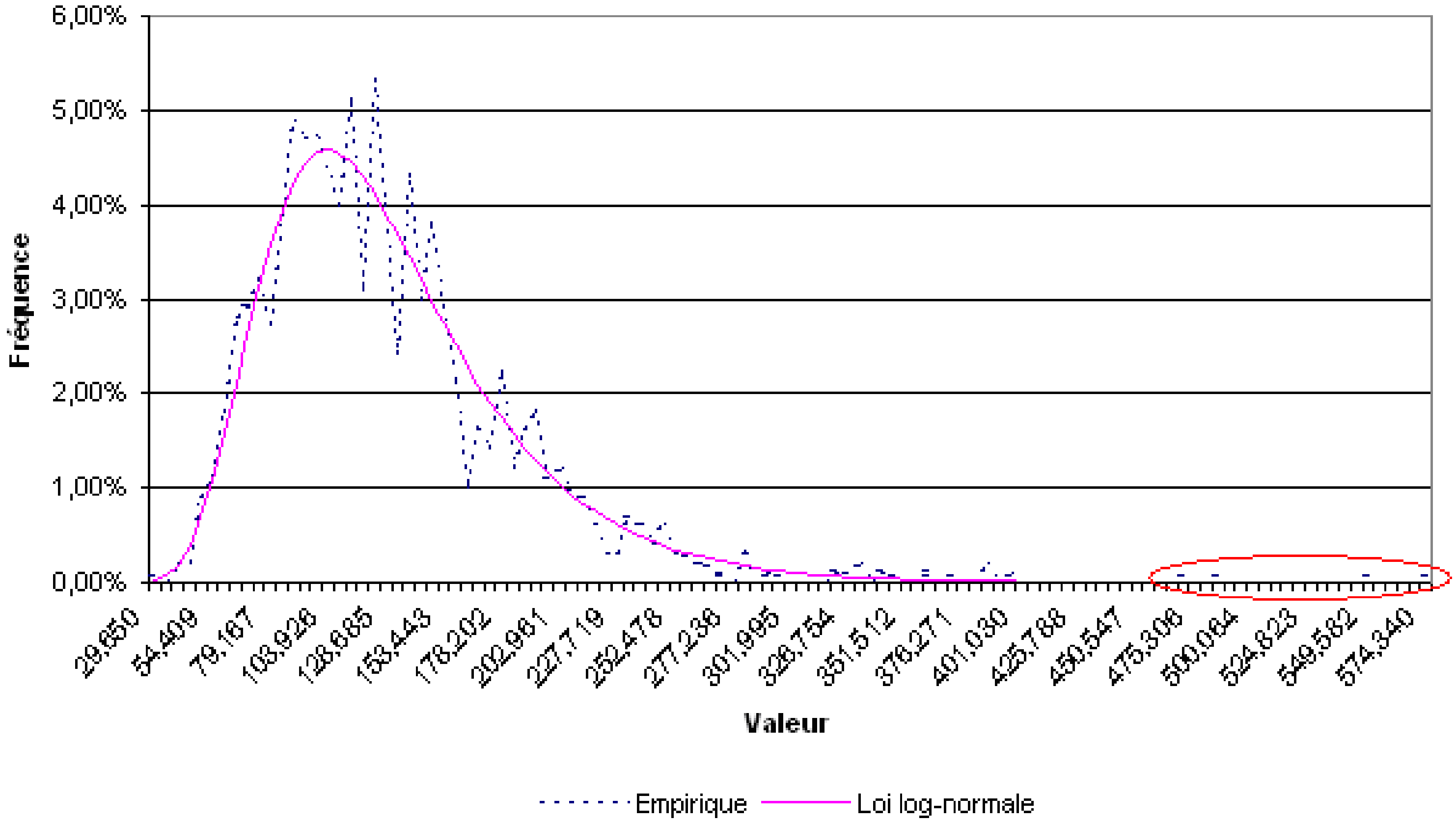
This presentation is inspired by work in which we had to estimate high level expected shortfall of a claim distribution.

Introduction

In order to estimate the expected shortfall, we explored 3 ways:

- Estimation based on a parametric adjustment of the claim distribution;
- Estimation based on the extreme value theory;
- A compound approach which will be developed later in the presentation

1. Parametric approach



Parametric approach often underestimates the tail of the distribution.

1. Parametric approach

Identification of extreme values

We fix a high-order percentile and compare the theoretical and the observed number of values which surpasses the chosen percentile.

In our example, with *p* = 99.8% and for a sample of 1 000 values, we observe 5 values which exceed the percentile.

If we consider the number of values exceeding a threshold *u* is approximately normal, we get a test to reject or not the adjustment :

$$P\left(N_u \geq k\right) \approx 1 - \phi\left(\frac{k - nS(u)}{\sqrt{nS(u)\left(1 - S(u)\right)}}\right)$$

2. Extreme value theory inputs

1. Main results

Let's recall the Pickands-Balkema-de Haan theorem:

$$F \in DA\left(G_{\xi}\right) \Leftrightarrow \exists \beta(.) > 0, \lim_{u \to x} \sup_{x>0} \left\{\left|F_u(x) - H_{\xi,\beta(u)}(x)\right|\right\} = 0$$

where :

- $F_u(x) = \Pr\left[X - u \leq x \mid X > u\right]$
- $H_{\xi,\beta(u)}$ denotes the Generalised Pareto Distribution (GPD).

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2. Extreme value theory inputs

1. Main results

A traditional approach is the Peaks-Over-Threshold (POT) method which consists in fitting the extreme values of the data with the GPD in order to compute the following results:

If $\xi<1$ and $Y \sim GPD(\xi,\beta)$ we have:

- $\mathrm{E}\left(1+\frac{\xi}{\beta}Y\right)^{-r} = \frac{1}{1+\xi r}$ for $r > -\frac{1}{\zeta}$
- $\mathrm{E}\left[\ln\left(1+\frac{\xi}{\beta}Y\right)\right]^{k} = \xi^{k} k!$ for $k \in \mathbb{N}$

2. Extreme value theory inputs

1. Main results

- $$\mathrm{E}\left[Y\left(\bar{H}_{\xi,\beta}(Y)\right)^{r}\right]=\frac{\beta}{(r+1-\xi)(r+1)}$$ for $(r+1)/\xi>0$
- $$\mathrm{E}\left[Y^{r}\right]=\frac{\beta^{r}\Gamma\left(\xi^{-1}-r\right)}{\xi^{r+1}\Gamma\left(\xi^{-1}+1\right)}r!$$ for $r\leq\left[\xi^{-1}\right]$
- $$\mathrm{E}[Y]=\frac{\beta}{1-\xi}$$ for $\xi<1$

If $\xi>1$, the expectation is not finite.

2. Extreme value theory inputs

2. Calibration of the GPD

In order to calibrate the GPD, you have to:

- determinate the level of the threshold;
- compute a maximum-likelihood estimation on the costs which surpass the threshold :

$$\begin{cases} \xi = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \ln\left(1+\tau X_i\right) =: \hat{\xi}(\tau), \\ \dfrac{1}{\tau} = \dfrac{1}{n}\left(\dfrac{1}{\xi}+1\right)\displaystyle\sum_{i=1}^{n} \dfrac{X_i}{1+\tau X_i}. \end{cases}$$

where $\tau = \xi / \beta$

2. Extreme value theory inputs

2. Calibration of the GPD

When ξ>-1/2, Hosking and Wallis (1987) demonstrated the asymptotic normality of the maximum-likelihood estimator:

$$n^{1/2}\left(\hat{\xi}_n - \xi, \frac{\hat{\beta}_n}{\beta} - 1\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left[0, (1+\xi)\begin{pmatrix} 1+\xi & -1 \\ -1 & 2 \end{pmatrix}\right]$$

2. Extreme value theory inputs

3. Expected shortfall

We compare 3 estimators of $e(u) = \mathrm{E}\left[X - u \mid X > u\right]$

- empirical estimator: $\hat{e}_1(u) = \dfrac{1}{N_u} \sum_{j=1}^{n} (x_j - u) 1_{\{x>u\}}(x_j)$
- GPD based estimator: $\hat{e}_2(u) = \dfrac{\hat{\beta}(u)}{1-\hat{\xi}}$
- Hill based estimator (ξ>0): $\hat{e}_3(u) = \dfrac{\hat{\xi} u}{1-\hat{\xi}}$

2. Extreme value theory inputs

4. Which threshold?

It's one of the key point using extreme values theory.

A classical approach is to consider a plot with the expectation of the excess beyond a threshold and to choose the smallest one for which the points are aligned on a pent of $\xi/(1-\xi)$.

An alternative approach consists in computing the tail factor Hill estimator and to consider the smallest threshold for which the tail factor is near to be constant.

2. Extreme value theory inputs

4. Which threshold?

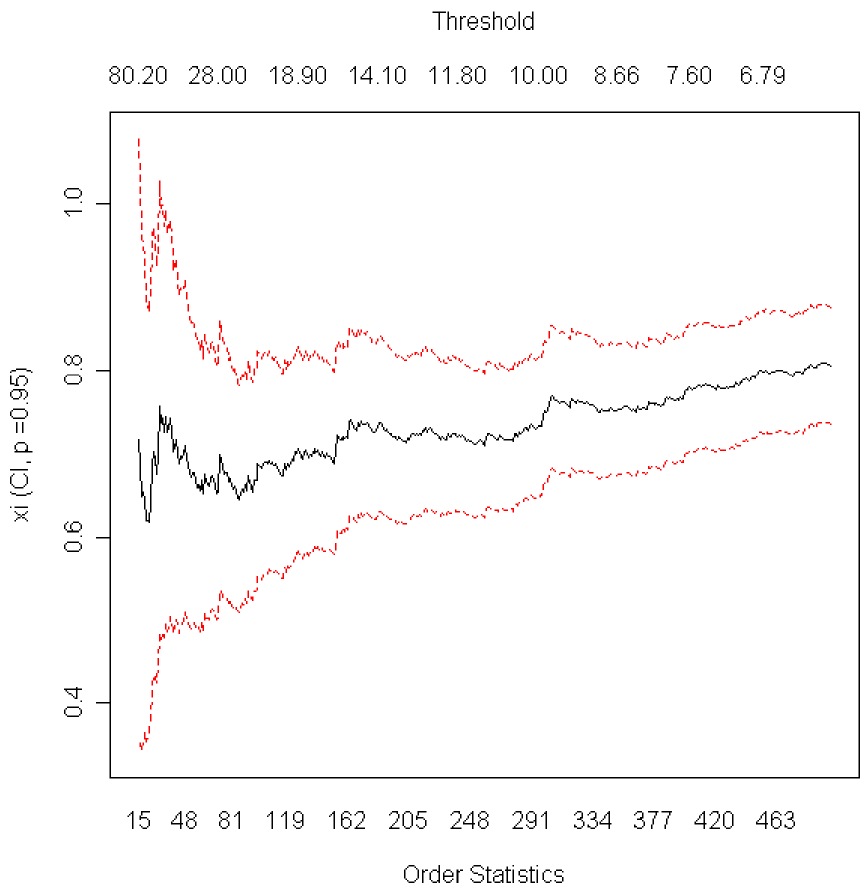
For the Hill estimator, De Haan and Peng (1998) proposed to used the number of observations which minimizes the quadratic error:

$$k^*(n) = \begin{cases} 1 + n^{2\xi/(2\xi+1)} \left(\frac{(1+\xi)^2}{2\xi} \right)^{1/(2\xi+1)}, & \text{si } \xi \in]0;1[\\ 2n^{2/3}, & \text{si } \xi > 1. \end{cases}$$

This expression needs as an input the tail factor.

2. Extreme value theory inputs

4. Which threshold?



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3. Illustration

1.1. Illustration: GPD – 10^6 simulations

	Estimateur GPD			Estimateur Hill		Estimateur empirique	Valeur théorique
Quantile	e_est	xi_est	beta_est	e_est	xi_est	e_est	e_th
90,0%	22,82	0,75	5,66	30,55	0,83	26,79	22,49
90,5%	23,70	0,75	5,88	32,03	0,83	27,87	23,38
91,0%	24,77	0,75	6,11	33,68	0,83	29,05	24,34
91,5%	25,90	0,75	6,37	35,48	0,83	30,35	25,41
92,0%	27,08	0,75	6,68	37,56	0,83	31,79	26,59
92,5%	28,49	0,75	7,00	39,95	0,83	33,39	27,91
93,0%	29,91	0,75	7,40	42,64	0,83	35,19	29,39
93,5%	32,02	0,76	7,76	45,79	0,83	37,22	31,07
94,0%	34,30	0,76	8,21	49,22	0,84	39,54	32,99
94,5%	36,22	0,76	8,82	53,10	0,84	42,21	35,22
95,0%	38,65	0,75	9,52	57,71	0,84	45,35	37,83
95,5%	41,45	0,75	10,37	63,08	0,84	49,08	40,94
96,0%	44,97	0,75	11,38	69,59	0,84	53,59	44,72
96,5%	49,88	0,75	12,55	78,22	0,84	59,20	49,43
97,0%	55,57	0,74	14,21	89,10	0,84	66,37	55,49
97,5%	63,05	0,74	16,29	103,73	0,84	75,94	63,62
98,0%	73,62	0,74	19,32	124,31	0,84	89,47	75,21
98,5%	90,40	0,73	24,02	156,94	0,84	110,38	93,32
99,0%	119,43	0,73	32,73	216,18	0,84	148,05	126,49
99,5%	200,26	0,73	53,76	371,16	0,84	242,86	212,73

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3. Illustration

1.2. Illustration: GPD – 5 000 simulations

	Estimateur GPD			Estimateur Hill		Estimateur empirique	Valeur théorique
Quantile	e_est	xi_est	beta_est	e_est	xi_est	e_est	e_th
90,0%	21,33	0,78	4,63	20,56	0,78	29,46	22,49
90,5%	23,60	0,80	4,66	21,70	0,78	30,70	23,38
91,0%	24,83	0,80	4,85	22,18	0,78	32,06	24,34
91,5%	32,37	0,86	4,66	23,54	0,78	33,56	25,41
92,0%	41,01	0,89	4,68	25,04	0,78	35,23	26,59
92,5%	39,54	0,87	5,07	25,95	0,78	37,09	27,91
93,0%	56,79	0,91	5,06	27,68	0,78	39,20	29,39
93,5%	101,55	0,95	5,09	29,85	0,78	41,58	31,07
94,0%	87,50	0,94	5,62	31,26	0,78	44,33	32,99
94,5%	85,19	0,93	6,16	33,58	0,78	47,55	35,22
95,0%	-707,40	1,01	5,96	36,21	0,78	51,33	37,83
95,5%	207,88	0,97	7,07	38,17	0,78	55,88	40,94
96,0%	-188,70	1,04	7,12	42,67	0,78	61,47	44,72
96,5%	-47,04	1,15	7,02	46,96	0,78	68,47	49,43
97,0%	-80,05	1,11	8,83	52,71	0,79	77,63	55,49
97,5%	455,07	0,97	13,12	58,45	0,79	90,23	63,62
98,0%	-237,98	1,06	14,13	74,28	0,79	108,57	75,21
98,5%	-133,02	1,13	17,46	94,40	0,80	137,47	93,32
99,0%	-234,57	1,12	28,61	125,14	0,79	192,11	126,49
99,5%	582,39	0,84	90,45	220,16	0,79	341,64	212,73

3. Illustration

1.3. Remarks

- The empirical estimator is not optimal.
- No absolute order between the GPD and Hill estimators
- The estimation of the tail factor is far to be obvious.

3. Illustration

2.1. Illustration: log-normal – 10^6 simulations

	Estimateur GPD			Estimateur Hill		Estimateur empirique	Valeur théorique
Quantile	e_est	xi_est	beta_est	e_est	xi_est	e_est	e_th
90,0%	2,82	0,26	2,09	3,13	0,46	6,42	2,81
90,5%	2,87	0,26	2,13	3,23	0,47	6,57	2,83
91,0%	2,91	0,26	2,17	3,33	0,47	6,72	2,85
91,5%	2,95	0,25	2,20	3,45	0,47	6,89	2,87
92,0%	3,00	0,25	2,24	3,57	0,47	7,07	2,90
92,5%	3,05	0,25	2,28	3,70	0,47	7,27	2,93
93,0%	3,11	0,25	2,33	3,85	0,47	7,48	2,96
93,5%	3,16	0,25	2,37	4,01	0,47	7,71	3,00
94,0%	3,23	0,25	2,43	4,19	0,47	7,96	3,04
94,5%	3,30	0,25	2,48	4,38	0,47	8,25	3,09
95,0%	3,38	0,25	2,55	4,59	0,47	8,57	3,15
95,5%	3,48	0,24	2,62	4,84	0,47	8,93	3,21
96,0%	3,58	0,24	2,72	5,12	0,47	9,34	3,29
96,5%	3,70	0,24	2,80	5,46	0,47	9,83	3,38
97,0%	3,84	0,24	2,92	5,85	0,47	10,41	3,49
97,5%	4,01	0,24	3,03	6,36	0,47	11,12	3,62
98,0%	4,24	0,24	3,24	6,98	0,47	12,04	3,80
98,5%	4,54	0,24	3,45	7,86	0,47	13,30	4,04
99,0%	5,00	0,23	3,83	9,19	0,47	15,23	4,40
99,5%	5,92	0,22	4,62	11,76	0,47	18,98	5,10

3. Illustration

2.2. Illustration: log-normal – 5 000 simulations

	Estimateur GPD			Estimateur Hill		Estimateur empirique	Valeur théorique
Quantile	e_est	xi_est	beta_est	e_est	xi_est	e_est	e_th
90,0%	2,52	0,20	2,02	2,78	0,44	6,09	2,81
90,5%	2,54	0,20	2,05	2,85	0,44	6,22	2,83
91,0%	2,58	0,19	2,09	2,91	0,44	6,36	2,85
91,5%	2,63	0,17	2,18	2,97	0,43	6,51	2,87
92,0%	2,66	0,17	2,21	3,09	0,44	6,67	2,90
92,5%	2,68	0,18	2,21	3,24	0,44	6,84	2,93
93,0%	2,70	0,19	2,19	3,38	0,44	7,02	2,96
93,5%	2,77	0,16	2,31	3,48	0,44	7,23	3,00
94,0%	2,81	0,16	2,35	3,67	0,44	7,45	3,04
94,5%	2,86	0,15	2,42	3,80	0,44	7,70	3,09
95,0%	2,85	0,19	2,32	4,04	0,44	7,97	3,15
95,5%	2,87	0,21	2,27	4,25	0,44	8,27	3,21
96,0%	2,96	0,20	2,35	4,43	0,44	8,61	3,29
96,5%	3,06	0,19	2,47	4,71	0,44	9,01	3,38
97,0%	3,19	0,16	2,66	4,99	0,44	9,49	3,49
97,5%	3,40	0,11	3,03	5,31	0,44	10,10	3,62
98,0%	3,44	0,13	2,99	5,91	0,44	10,85	3,80
98,5%	3,46	0,18	2,84	6,67	0,44	11,85	4,04
99,0%	3,68	0,20	2,93	7,63	0,44	13,30	4,40
99,5%	4,29	0,49	2,19	9,68	0,44	15,93	5,10

3. Illustration

2.3. Remarks

- The log-normal distribution belongs to the domain of attraction of Gumbel $\Rightarrow$ the Hill estimator couldn't be applied.
- GPD is the most relevant estimator but doesn't seem very robust.

4. A parametric compound approach

Using extreme value theory is delicate due to:

- the choice of the threshold;
- the estimation risk computing the extreme estimator.

We suggest an alternative approach based on a compound parametrical adjustment based on the GPD behaviour of the excess of the costs beyond an extreme threshold.

4. A parametric compound approach

1. Detail of the approach

Step 1: Parametric adjustment of the cost of claims

Step 2: Composition of the adjusted distribution with the Pareto distribution for the amounts which exceed a (unknown) threshold

Step 3: Calibration of the compound model (maximum-likelihood estimator)

Step 4: Calculation of the expected shortfall on the basis of the parameter of the Pareto Distribution.

4. A parametric compound approach

2. Example : log-normal / Pareto

The estimation of parameters can be performed by the maximum likelihood method:

$$l\left(x_1,..,x_n;\mu,\sigma,m,\alpha\right)=cste-\left(k-1\right)\ln\left(\sigma\right)-\frac{1}{2}\sum_{i=1}^{k-1}\left(\frac{\ln\left(x_{(i)}\right)-\mu}{\sigma}\right)^2$$

$$+\left(n-k+1\right)\ln\left(\alpha\right)+\alpha\left(n-k+1\right)\ln\left(m\right)-\alpha\sum_{i=k}^{n}\ln\left(x_{(i)}\right)+\left(n-k+1\right)S_0\left(m\right)$$

with $k=\min\left\{i\,;x_{(i)}\geq m\right\}$

4. A parametric compound approach

2. Example : log-normal / Pareto

$$\max_{(\mu,\sigma,m,\alpha)} l\left(x_1,\ldots,x_n;\mu,\sigma,m,\alpha\right) = \max_{m} \max_{(\mu,\sigma,\alpha)} l\left(x_1,\ldots,x_n;\mu,\sigma,m,\alpha\right)$$

$$\hat{\mu} = \frac{1}{k-1}\sum_{i=1}^{k-1} \ln x_{i,n} \qquad \hat{\sigma} = \sqrt{\frac{1}{k-1}\sum_{i=1}^{k-1}\left(\ln x_{i,n} - \hat{\mu}\right)} \qquad \hat{\alpha} = \frac{n-k+1}{\sum_{i=k}^{n} \ln\left(\frac{x_{i,n}}{m}\right)}$$

4. A parametric compound approach

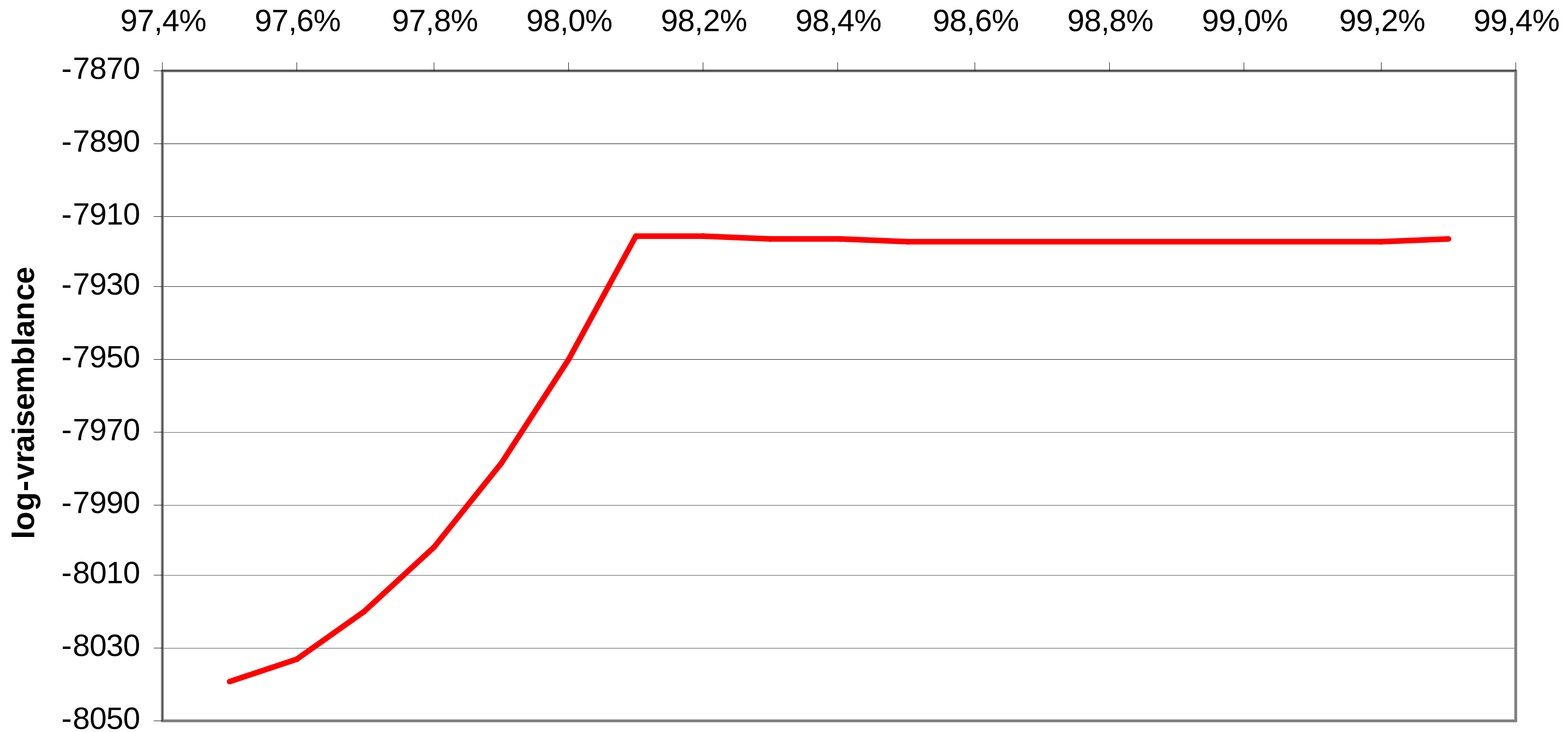
2. Example : log-normal / Pareto

It remains to eliminate *m*, unknown, in the above equation. In practice we can proceed in the following way:

- we fix *k* (while starting for example by $k = 95\% \times n$)
- we calculate $\hat{\mu}$ and $\hat{\sigma}$
- we calculate $\hat{m} = \exp\left(\hat{\mu} + \hat{\sigma}^{-1}\left(\frac{k}{n}\right)\right)$
- the estimator (pseudo maximum likelihood) of tail parameter is given by the expression: $$\hat{\alpha} = \frac{n-k+1}{\sum_{i=k}^{n} \ln\left(\frac{x_{(i)}}{\hat{m}}\right)}$$

4. A parametric compound approach

2. Example : log-normal / Pareto



4. A parametric compound approach

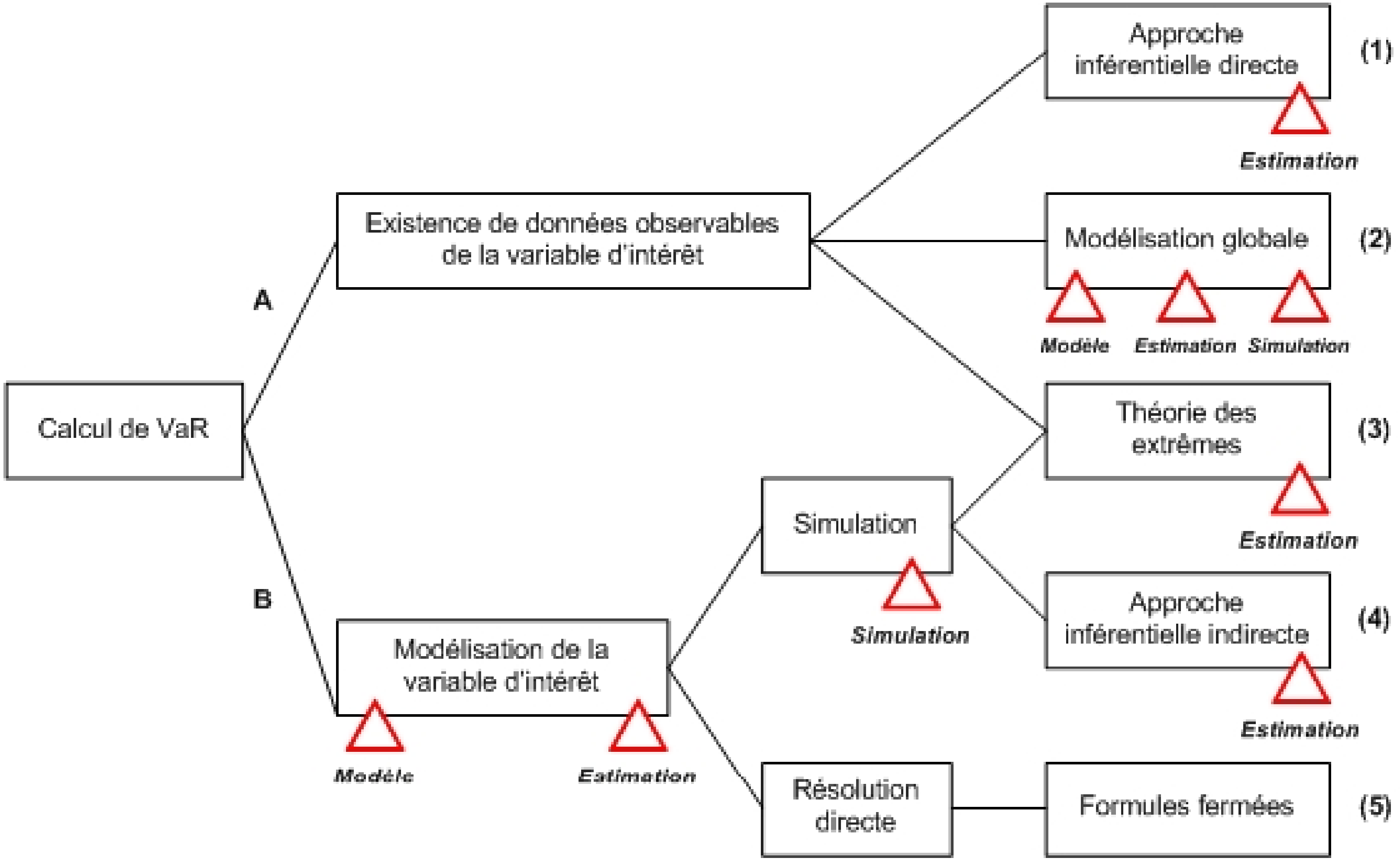
Conclusion

This alternative approach:

- enables to work with a parametric distribution;
- exchange the estimation of the threshold risk with a specification risk.

The approach was developed in an Expected Shortfall estimation but may be used in a VaR estimation.

Conclusion



Calcul de VaR
A
B
Existence de données observables de la variable d'intérêt
Modélisation de la variable d'intérêt
Modèle
Estimation
Simulation
Simulation
Résolution directe
Approche inférentielle directe
(1)
Estimation
Modélisation globale
(2)
Modèle
Estimation
Simulation
Théorie des extrêmes
(3)
Estimation
Approche inférentielle indirecte
(4)
Estimation
Formules fermées
(5)

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